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The Unity of Logic, Pedagogy and Foundations in Grassmann's Mathematical Work

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Hermann Grassmann's *Ausdehnungslehre* of 1844 and his *Lehrbuch der Arithmetik* of 1861 are landmark works in mathematics; the former not only developed new mathematical fields but also both contributed to the setting of modern standards of rigor. Their very modernity, however, may obscure features of Grassmann's view of the foundations of mathematics that were not adopted since. Grassmann gave a key role to the learning of mathematics that affected his method of presentation, including his emphasis on making initial assumptions explicit. In order to better understand this less well-known aspect of his work it will help to examine why some commentators have overlooked his theme of unifying logic, pedagogy and foundations, while others have recognized it.

1. Introduction

When the German mathematician Hermann Grassmann (1809-1877) published his first and most important work, the *Ausdehnungslehre*, the 'theory of extension', (*1844*), it went virtually unrecognised, even when he presented essentially the same results in an entirely different way in his revised version, *1862.* It took almost another decade before the relevance of Grassmann's discoveries to mathematics at large began to be recognized. This may have been due partly to their large scale and novelty. He intended this to be a 'new branch of mathematics, explained through applications to the other branches of mathematics as well as to statics, mechanics, the theory of magnetism, and crystallography', as the subtitle proclaimed.

Grassmann was the first mathematician to explicitly make a distinction between geometry, as the science of our physical space, and a purely mathematical treatment of abstract objects (which he termed 'extensive magnitudes') that would have geometry as one of its applications but which would not be limited to three dimensions. Without going into details here, we can already see something of Grassmann's grand vision and his methodology in his explanation of the place of the *Ausdehnungslehre* in mathematics. He conceived mathematics as consisting of four branches based on two types of elements (equal and different) and two modes of generation of those elements (continuous and discrete). For example, in number theory or arithmetic the objects of study have an algebraic-discrete form and are generated from equal elements by a discrete positing and connecting, as in 1, 1 + 1, …. . In this classification scheme, the theory of extension is the branch concerned with objects of a combinatorial-continuous form generated from different elements by a single continuous mode. Grassmann's primary concrete example of an extensive magnitude is the bounded line segment conceived as generated from the points which constitute it. (Grassmann carefully emphasized that such a geometrical example was only a concrete representation of an extensive magnitude and not an example of the abstract entity.) The other two branches of mathematics are combinatorics (different elements, discrete generation) and analysis (equal elements, continuous generation). This overview of pure mathematics is followed in *1844* by a derivation of the concept of the theory of extension and by a discussion of the form of presentation (which latter will be addressed below). This is followed by a section which might be described today as devoted to a universal algebra and which Grassmann called the general theory of forms (*Formenlehre*), that is, 'truths, which relate to all branches of mathematics in the same way and thus assume only the general concepts of equality and difference, and of connection and separation' (*1844,* §1).[1]

Perhaps mathematical developments needed to catch up with him, but certainly during most of his life the blame for the slow recognition was given not only to the novelty of the ideas but to the style of presentation, especially in the case of the *Ausdehnungslehre* of 1844. This earlier version has often been described as a work that was overly 'philosophical'. Indeed Grassmann wrote in the foreword to his *1862* that, of the few mathematicians who shared their reactions to *1844* with him, all thought the work 'more philosophical than mathematical'. He does not name anyone but we know that one of his staunchest supporters, and probably the most famous one, the German mathematician A. Möbius, in 1845 declined to review his *Ausdehnungslehre* for publication pleading as follows:

> I am not competent to appreciate, indeed to even understand properly, the philosophical element of your excellent writing, forming the basis as it does of the mathematical element. I have come to realize also that, after a number of attempts to study your work without interruption, I have in each case been stopped by the great philosophical generality. (*Grassmann, H. G. 1894-1911,* vol. II, part 2, 100)

This seems particularly ironic since Grassmann anticipated such criticism and took care to try to prepare his readers when he wrote that among mathematicians there was, 'not without some justification, a certain aversion towards philosophical discussion of mathematical and physical matters; and indeed most investigations of this kind, as they are conducted by Hegel and his school in particular, suffer from an unclearness and arbitrariness which negates all their results' (*1844,* p. xv). In this mathematical context there may be a difference of usage of the term 'philosophy' between Grassmann and his contemporaries, on the one hand, and later students of the subject on the other. A modern reader would be apt to take this one mention of Hegel as the only explicitly philosophical reference Grassmann made in *1844,* and find that nowhere else in his extant writings did he relate his mathematical work to a philosopher or to a philosophical school. Furthermore, to such a modern reader, none of those who knew Grassmann, or who later in the nineteenth century propagated his work, suggested any such connection in their writings. His biographer and principal editor of his collected mathematical and physical

[1] References are made to the sections of the *Ausdehnungslehre* of 1844 in order to accommodate the various printings and translations.

works, Friedrich Engel, quoted from an early curriculum vitae where Grassmann credited the theologian and philosopher F. Schleiermacher, whose lectures Grassmann attended at Berlin University from 1827 to 1830, with being a strong influence in all aspects of his thinking (*Grassmann, H. G. 1894-1911*, vol. III, part 2, pp. 20-22). According to Engel, Grassmann continued reading Schleiermacher throughout his life. The specific connection of Schleiermacher to Grassmann's mathematics, however, seems not to have been made until Arthur Schweitzer in *1915* (see Section 3 below). In the context of the meanings of 'philosophy' and 'logic' in the *Ausdehnungslehre* (quoted in Section 2 below). Grassmann regarded the philosophical component of the work to be the part devoted to logic. Thus, when Grassmann and his mathematical contemporaries refer to the 'philosophical' nature of the *Ausdehnungslehre,* they appear to be referring largely to what we would today call the abstract mathematical foundation which Grassmann laid down in *1844*. In his 1862 rewriting, designed to accommodate the difficulties readers had with the 'philosophical' nature, by far his largest change with respect to content was the omission of all of the discussion about the division of mathematics, the theory of forms, and the method of presentation. Some commentators who have not read the *Ausdehnungslehre* have perhaps misunderstood these early judgements by mathematicians and assumed that they referred to mathematically irrelevant matters. Morris Kline, for example, pictured its ideas as 'shrouded … with mystic doctrines' (*Kline 1972*, 782). By contrast, as we shall see in Section 5, mathematicians today are in a better position to appreciate Grassmann's work as a whole.

In addition to the logical basis in Grassmann's sense, his idea of a sound foundation for mathematics was designed to support both the method (or at least the course) of discovery and the method of proof. This comes from a leading feature in Grassmann's conception of foundation: the essential role that the learning of mathematics plays. That there could be a pedagogical motivation for combining threads of discovery and of proof in a mathematical presentation would hardly be unusual, but in Grassmann's case it appears to be much more than a device to aid the reader; he appears to regard the pedagogical involvement as an essential part of the justification of mathematics as a science. Since the main influence of the *Ausdehnungslehre* has in effect been splintered off into mathematics and logic, the historical understanding of Grassmann's original, unified work requires a somewhat different approach from that usually given in the general histories of those subjects.

Though Grassmann has always held a respectably prominent place in general histories of mathematics and, to a lesser extent, logic, only in recent years has there been a significant number of papers and books devoted to him, including new translations into French and English (cited below). New works have appeared treating the history of logic and its relation to the foundations of mathematics that include Grassmann. Linear algebra, which began with Grassmann, has assumed an increasingly important role in the mathematical curriculum, with a consequent increase of interest in its origins. The present paper draws upon some newer literature that relates to this unifying theme either directly or indirectly. Since the *Ausdehnungslehre* and its poor initial reception can be taken as a cautionary tale for later writers, it is no surprise that since then it is not easy to find a work in this area of mathematics whose theme can match it with respect to grand mathematical, philosophical and pedagogical intentions. Nevertheless it is interesting how and to what extent investigations in the intersections of these areas still form a part of Grassmann's legacy.

Thus this survey will look at several works in mathematics, philosophy and pedagogy, including works in the history of these subjects. The main focus is on recent decades, but first a brief indication is given of Grassmann's foundational ideas, followed by a review of some earlier works that will set the stage and help to show to what extent the more recent work is new, rather than simply a rediscovery or rephrasing of what was done before.

2. Learner-based presentation and the role of logic

In the foreword to *1862,* his reworking of the little-read *1844*, Grassmann recognized the desirability of having an appreciative readership and expressed a willingness to try to achieve this by presenting only the bare mathematical results while still being as rigorous as possible. His very brief explanation of what was lost from his original program by making such a radical change in his presentation serves to introduce the subject of this paper: Grassmann's unified view of logic, pedagogy and foundations.

> This difficulty [which readers of *1844* had] could not, however, be removed without substantially changing the plan of the whole work. For it is not inherent in some arbitrarily chosen form but rather in the plan that I envisaged: to build the science from the ground up, independently of the other branches of mathematics. The direct implementation of this plan raises significant difficulties for that form of presentation if the implementation is to be as expeditious for the science itself as it becomes subjectively [for the reader]. This is particularly so in a science such as the theory of extension which extends and intellectualizes the sensual intuitions of geometry into general, logical concepts and which, with respect to abstract generality, is not simply one among the other branches, such as algebra, combination theory, and function theory, but rather far surpasses them by unifying all of their fundamental elements. It thus could be said to form the keystone of the entire structure of mathematics.
>
> I therefore had to abandon this whole plan, and now in the present work have assumed the other branches of mathematics, at least in their elementary development. Also, with respect to the form of presentation, I have adopted exactly the opposite direction from the earlier version since here I have applied what is generally speaking [*überhaupt*] the most rigorous mathematical form that we know, the Euclidean, and have relegated to the Remarks all that serves in illustrating [*Erläuterung*] or motivating [*Begründung*] the chosen course.
>
> … the new method is in no way to be preferred over the older. On the contrary, the method of the first edition probes more deeply into the essence of the subject and thus from a purely scientific point of view has a decided advantage compared with the newer method. On the other hand this latter method is more acceptable to

> those mathematicians who are not willing to see lay fallow mathematical treasures that are obtainable in some alternate way, and it is easier for them to understand in any case. Thus the two presentations complement and elucidate each other. (*Grassmann, H. G. 1862*, pp. iii-iv)

In addition to introducing a new branch of mathematics in *1844*, we see here that Grassmann had also hoped to introduce a rather novel method of presentation designed to meet the highest standards of rigor while also making clear the original motivations behind the new concepts he had discovered. Nowhere does he directly discuss the meaning of 'rigor' or what constitutes a proof but the main clue to Grassmann's understanding of the current state of such foundational affairs in mathematics, and of his own contribution to changing that status quo, comes from the reluctant way he adopts what he regarded as a more 'mathematical', Euclidean style of axiom-definition-theorem-proof presentation in *1862*. In this work he assumed from the beginning a number system with the usual arithmetic properties and thus bypassed all of the introductory, foundational work of *1844*. Part of his goal in *1844* was to clarify the true starting points for mathematics as a whole and rid it of hidden assumptions, and from a modern point of view his reluctance to simply try to employ the *1862* mode of presentation for the whole of his work may seem puzzling. However, the Euclidean style did have its critics and Grassmann hoped to capitalize on that fact. As he argued in *1844*:

> 13. The essence of the philosophical method is that it proceeds by means of contrasts to arrive at the particular from the general: the mathematical method, on the other hand, proceeds from the simplest concepts to the more complex, and thus, through connecting of the particular, attains new and more general concepts. …
>
> 14. Since both mathematics and philosophy are sciences in the strictest sense, so must the methods in both have something in common which makes them thus scientific. Now, we add the scientific quality [*Wissenschaftlichkeit*] to a method of treatment when the reader is, on the one hand, led by it necessarily to the recognition of each individual truth, and, on the other, is put in the position at each point of the development of seeing the direction of further progress.
>
> The indispensability of the first requirement, namely scientific rigor, anyone will grant. As for the second, that is another matter, not yet properly recognized by most mathematicians. Proofs often occur in which at first, if it were not for the statement [of the theorem] standing above, one would have no idea to where it is supposed to lead. Consequently, after one has blindly and haphazardly followed each step for quite some time, finally, before you expect it, that truth which was to be proven is suddenly attained. Such a proof can perhaps leave nothing to be desired in rigor, but it is not scientific; the second requirement is lacking, the provision of an overview. Whoever enters into such a proof does not attain an enlightening understanding of the truth, but rather remains in entire dependence on the particular manner in which the truth was found, unless subsequently he creates that overview himself. And this sense of constraint [*Unfreiheit*] which

arises in such a case, at least while learning it, is a most oppressing thing for one used to thinking freely and independently, and mastering all which he assumes spontaneously and vitally. If, on the other hand, the reader is put in the position at each point of the development of seeing where he is going, then he remains in command of the material, he is no longer bound to the particular form of presentation, and the incorporation becomes a true reproduction.

15. At each point of the development the manner of further development is determined essentially through a leading idea [*leitende Idee*], which is either nothing other than a conjectured analogy with related and already known branches of knowledge, or which---and this is the best case---is a direct presentiment [*Ahnung*] of the next truth to be sought. …

16. Thus the scientific presentation in essence is an interlocking of two series of developments, of which one consistently leads from one truth to another and makes up the essential content, while the other governs the process itself and determines the form. In mathematics both these series stand apart from each other in the sharpest way.

It has been the practice in mathematics for a long time, and Euclid himself set the precedent, to allow only that one series of development to predominate which forms the essential content; as for the other, it was left for the reader to make it out between the lines. However complete the arrangement and presentation of that sequence of development may be, it is still impossible for the one who is supposed for the first time to learn to know the science, to readily obtain through that one series an overview at each point of the development and to put himself in the position of continuing further independently and freely. To this end it is much more essential that the reader be placed as far as possible in that position in which the discoverer of the truth had to be in the most favourable case. …

… [M]ore recent mathematicians, and in particular the French, have begun to interweave both series of development.

… [I]nherent in the second series of development is a quite opposite character to the first, and the interpenetration of the two appears more difficult than in any other science. One ought not, on account of this difficulty---as frequently happens with German mathematicians---give up and repudiate the whole procedure.

In the present work I have thus taken the way described and it seems to me this is all the more necessary for a new science where its ideas should come to light at the same time at the beginning. (*Grassmann, H. G. 1844,* Introduction)

In calling for the reader to have the same vantage point as the discoverer it should be noted that Grassmann adds that it is the discoverer 'in the most favourable case'. Grassmann might have agreed in some sense with Gauss's opinion that exposing to public view the work done in discovering new results in mathematics was like leaving the scaffolding up around a finished cathedral. Grassmann's notion, that it was important to present mathematics in a fashion that makes clear why it was constructed the way it was, does not entail, however, exposing the conjectures, false trails, or mistakes that Gauss may have had in mind as making up the scaffolding. Instead Grassmann's *Ausdehnungslehre* can be seen as another effort in the line of attempts to combine a rigorous method of proof with a method that aids discovery. From this point of view the two methods have their roots in ancient mathematics, and are often expressed as the 'analytic' and the 'synthetic', though these terms have such a convoluted history that deciding which term describes which method can depend on the particular context of the discussion. The meaning of the terms most relevant for Grassmann's methods was used by the Greek mathematician Pappos who described an 'analytic' heuristic method, where the desired proposition was assumed and inquiry proceeded from it to find propositions already proven, and to a 'synthetic' method which ordered the consequences of already proven propositions to arrive at the desired one.[2] Grassmann only used the terms in his general theory of forms where he discussed connection [*Verknüpfung*] and separation [*Sonderung*]:

> The analytic process consists in seeking one member of a connection in terms of the result and the other member. … Since this analytic process can also be regarded as a connection, we distinguish the original or <u>synthetic</u> connection from the inverse or <u>analytic</u> connection. (*1844*, §5)

The two strands relate also to the sentiment, often expressed by mathematicians and logicians, that the purely formal, axiomatic expression of a body of mathematics does not capture the essence of the subject. The twentieth-century British mathematician G. H. Hardy concluded that 'there is strictly speaking no such thing as mathematical proof; … we can, in the last analysis, do nothing but point'. He draws a rough analogy with a teacher who sees a mathematical construct clearly, in the way that one might see a distant peak clearly: 'If he wishes someone else to see it, he <u>points to it</u>, either directly or through the chain of summits which led him to recognise it himself. When his pupil also sees it, the research, the argument, the <u>proof</u> is finished' (*Hardy 1929,* 18).[3]

It is no coincidence that the use of contrasting or complementary opposites appears rather prominently in the parts of Grassmann's work described thus far: philosophy (the general) and mathematics (the specific); the division of mathematics into four branches corresponding to combinations of two pairs of opposites, equal and

[2] F. Vieta in the sixteenth century linked the term 'analytic' to a whole branch of mathematics, namely algebra. Though Lagrange echoed this use in his *Méchanique analitique* of 1788, his proofs were synthetic in the sense used in Pappos's heuristics. More of this tortuous history is provided in *Grattan-Guinness 1997*.

[3] Another example of this sentiment is: 'The conclusion is that Logic, conceived as an adequate analysis of the advance of thought, is a fake. It is a superb instrument, but it requires a background of common sense' (*Whitehead 1941,* 700). More examples can be found in *Kline 1980*.

different elements, continuous and discrete modes of generation; connection and separation; analytic and synthetic connections; and, overarching all of these, the complementary strands making up the method of presentation. This feature permeates the whole work. Nevertheless this dialectical character is not discussed by Grassmann—the only mention of dialetic in the *Ausdehnungslehre* is in the quote given below concerning the nature of the formal sciences. Though it may have been an obvious feature to his readers, it appears to have been first analyzed in *Lewis 1977* where the connection is made with Schleiermacher's *Dialektik* of 1839 which Grassmann and his brother Robert read together in 1840.[4] This brand of dialectic differed from the better-known one of Hegel in a number of respects but the most relevant difference for the *Ausdehnungslehre* is that, for Grassmann and Schleiermacher, there was in general no synthesis or resolution of opposites. It was the tension between contrasting pairs that determined the concepts and the species-genus relationships between those concepts as in the division of mathematics into its branches. The poles that determine a contrasting pair are not absolutes. For example, the analytic process was introduced above as a kind of separation, but Grassmann indicated that it could also be conceived as a connection and this led in turn to the distinction between analytic and synthetic connections. For Schleiermacher there was one ultimate resolution of absolute identity and absolute diversity grounded in the unity of the actual world (*Schleiermacher 1839*, 309). Grassmann's implicit use of dialectic did not extend that far and his largest unifying notion is, as we have seen, an appeal to the 'scientific quality' that comes through his method of presentation. Nevertheless it seems clear that dialectic played a foundational role for Grassmann comparable to that which logic plays today in mathematics. In fact Grassmann identified logic and dialectic at the beginning of his Introduction to *1844*:

> 1. The highest division of all the sciences is into the real and the formal, where the first represent in thought the existent as standing independently over against thought, and have their truth in the correspondence of thought with that existent. The formal sciences, on the other hand, have as their object that which is posited through thought itself and have their truth in the correspondence of the reasoning processes among themselves. …
>
> 2. The formal sciences consider either the <u>general</u> principles of thought or they consider the <u>particular</u> which is posited through thought—the former is dialectic (logic), the latter pure mathematics.
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> The contrast between the general and the particular thus implies the division of the formal sciences into dialectic and mathematics. The first is a philosophical science since it searches for the unity in all thought, while mathematics takes the opposite direction since it conceives each thought individually as a particular.

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[4] The author's 1977 paper was an excerpt from his University of Texas at Austin dissertation which Ivor Grattan-Guinness encouraged him to publish.

> 3. Pure mathematics is therefore the science of the particular existent as something which has come to be through thought. The particular existent conceived in this sense we term a thought-form or, in short, a form. Thus pure mathematics is the theory of forms. (*1844*)

In the 1878 reprinting of *1844,* the author appended a footnote to the word 'logic' in the above paragraph 2: 'Logic presents a purely mathematical side, which can be termed formal logic, and I and my brother, Robert, have worked on this together' (*Grassmann, H. G. 1878*).[5] Grassmann provides a reference to the book *Grassmann, R. 1872*. This illustrates how, in a dialectical fashion, each division of the sciences reflects within aspects of itself each of its determining contrasts. Thus logic contains formal logic as the 'particular' aspect of logic, while mathematics contains the theory of forms as the 'general' aspect of mathematics. Logic and mathematics thus overlap in these areas.

To give some idea of how Grassmann's method of presentation played out in practice in the body of his *Ausdehnungslehre* is a challenge to do in one or two paragraphs. Grassmann's treatment of addition and subtraction of extensive magnitudes of the first step or order (called *Strecke* or stretches) took up fifteen pages, not counting illustrative applications. Their multiplication took twelve, but, if addition can be taken as a given operation, multiplication may be a more manageable topic for this illustration. For present purposes a sufficient idea of addition of stretches can given by the following example of adding linear motions: if a motion from point A to point B is followed by one in a different direction from B to C, the sum of the two motions is the motion from A to C, i.e. following what is today called the parallelogram rule for addition of vectors. The introduction of multiplication, as the next higher-order operation, in a dialectical fashion helps to explain and justify the earlier definition of addition of stretches. Multiplication as a general notion had already been introduced in the preliminary section on the theory of forms where its key defining property was distributivity over addition (on the right and the left). Grassmann began his discussion of multiplication of stretches with a geometrical argument where the stretch can be understood as a line segment:

> We start from geometry, not only in order to obtain the analogy according to which the abstract science must proceed, but also in order to have an intuitive idea in view which leads us through the unknown and often difficult course of the abstract development. We proceed from the stretch as a line segment to a spatial

[5] The two brothers collaborated rather closely on both the *Ausdehnungslehre* and the *Lehrbuch der Arithmetik* (*1861*) and, though it is not known exactly what each contributed, judging from his later work Robert, the younger broether, could well have been a major contributor within the philosophical and logical components. A brief account of Robert Grassmann's place in the history of logic is given in *Grattan-Guinness 1996* and *Grattan-Guinness 2000*. The collaboration of the brothers is discussed in *Schubring 1996a.*

> entity of higher order if we allow the entire stretch, i.e. each of its points, to describe a new stretch not homogeneous [that is, not in a continuation of the same line] to the first, so that all the points construct equal stretches. The surface thus generated has the form of a spar-crystal face (parallelogram). Now we give two such surfaces which belong to the same plane equal signs if, in passing from the direction of the stretch which is moved to the direction of the one constructed by the motion, in both cases it is towards the same side (e.g. to the left for both surfaces): they will be given different signs if such passage is in opposite directions. (*1844* §28)

Next Grassmann proved the following two theorems which are the geometrical forms of the left and right distributive properties:

> If in the plane a stretch is shifted by arbitrary stretches one after the other, then the entire surface described by this (giving signs to the individual surface parts in the specified way) has the same magnitude as if it were shifted by the sum of those stretches. …
>
> The surface which a broken line describes when it is placed in the plane is equal to that described by a straight line which has the same initial and terminal points. (*1844* §28)

Grassmann then used the general sign of a synthetic connection, introduced in the theory of forms, to temporarily write the result of shifting one stretch, *a*, by another, *b,* as $a \cap b$. The two theorems can then be expressed as:

$$a \cap (b + c) = a \cap b + a \cap c,$$

and

$$(b + c) \cap a = b \cap a + c \cap a.$$

He continues:

> These were, according to para 9 [general theory of forms], the relations which determine a connection as multiplicative. The particular characteristic of this multiplication and the type of signs and terminology founded on it we will provide in the rigorous scientific presentation. (*1844* §29)

At this point Grassmann delayed the ‘rigorous’ presentation of this type of multiplication in order to underline what he regarded as the significant confirmation multiplication gave of the definition of addition of stretches. Grassmann was anticipating that the reader might not find this idea of addition intuitively appealing. Using the illustration given above for adding two motions, in what sense can this be addition when the result is a different object that does not contain the addends? The result will have a different direction and its length will not be the sum of the lengths of the addends.

> In the relation presented here lies the most elegant justification of the concept of addition which we presented in the previous chapter.

> In fact, suppose one has an equation whose terms are stretches in the same plane but of unequal direction, and the equation no longer holds if the stretches are replaced by their lengths, thereby making the equation algebraic. We could then remove this apparent disharmony between geometric and algebraic equations immediately if we shift the whole system of those stretches within the same plane and introduce the resulting surfaces into the equation; in other words, if we multiply the equation by a stretch of the same plane. As we have just shown, the equation under discussion holds also algebraically for the resulting surfaces, as long as one observes the given sign principle. (*1844* §30)

After showing that essentially the same argument can be used even if the stretches are not in the same plane, Grassmann proceeded to give a formal development of multiplication. He pointed out the noteworthy property that the multiplication of stretches gives an element of a new species (Gattung) whereas addition resulted in an element of the same species. For example, the addition of two line segments resulted in a line segment, their multiplication resulted in a signed parallelogram. Thus multiplication (at least this type of multiplication, which Grassmann called outer multiplication) generates a higher-order element. In the formal development the primitive concept is 'generation' and a new mode of generation provides a new connection between elements. The argument closely parallels the geometric analogy where, in the expression $a \cap b$, a can be taken as the generator and b the measure of generation. The result is an extension of second order which can in turn undergo a generation of the same mode, c, and so forth, $a \cap b \cap c \ldots$ to obtain an extension of nth step. He then argued that 'if A and A_1 are two homogeneous [i.e. generated by the same mode] extensive magnitudes of arbitrary order and, further, generated in the same sense, and b represents a stretch then we always have:

$$(A + A_1) \cap b = A \cap b + A_1 \cap b,$$

where $A \cap b$ and $A_1 \cap b$ are homogeneous and where the sign of connection represents the new type of connection'. A similar argument shows that

$$A \cap (b + b_1) = A \cap b + A \cap b_1 ,$$

where b and b_1 are stretches generated in the same sense.

> It is clear that this can be extended, by repeated application of this principle of relation, to arbitrarily many factors. Since, according to §9 [general theory of forms] this principle is the fundamental principle of multiplication, then we will say that the new type of connection has the multiplicative relation to addition of that which is generated in an equal sense. Hence all the principles deduced from this hold here …. With this our new connection is now established, according to §12 [the real concept of multiplication in the general theory of forms], as multiplication and we thus introduce for it also the multiplication symbol [i.e. a dot]. (*1844,* §32)

Grassmann proceeded next to a more general notion of multiplication which distributes over addition of inhomogeneous stretches. His final stage was the proof of the characteristic properties of this new multiplication, called outer multiplication:

$$(a + b_1) \,.\, b = a \,.\, b \quad \text{and} \quad b \,.\, (a + b_1) = b \,.\, a$$

where b and b_1 are homogeneous stretches, a an arbitrary stretch, and the dot (introduced later in the *Ausdehnungslehre*) represents outer multiplication. From these and the fact that $(a + b) \,.\, (a + b) = 0$ he showed that $a \,.\, b + b \,.\, a = 0$ or $a \,.\, b = - b \,.\, a$. The fact that this multiplication is non-commutative, and thus unlike the usual arithmetic multiplication, is probably one of the reasons Grassmann took pains to justify it as fully as possible. It is also quite likely one of the reasons why some readers at the time—even among those who were willing to tackle the foundational component—found the whole of Grassmann's program objectionable. This unusual multiplication, and indeed all that preceded it in the book, appeared to depart from the prevailing Kantian notion of the indispensability of intuition in mathematics. E. F. Apelt, professor of philosophy at Jena, wrote to Möbius in 1845:

> Have you read Grassmann's remarkable *Ausdehnungslehre*? It seems to me that a false philosophy of mathematics is at the bottom of it. The essential character of mathematical knowledge, that it is intuitive, seems to be excluded from it completely. An abstract theory of extension such as Grassmann wishes can be developed only from concepts; but the source of mathematical knowledge is found not in concepts, but in intuition. (quoted from *Grassmann, H. G. 1894-1911*, vol. 3, part 2, 101).

3. Some early Grassmannians

Only when Grassmann's ideas began to be propagated by others, and by himself in journal articles, did either of his books devoted to the theory of extension receive much attention. It may be unprecedented in the history of science for such a seminal work to have appeared in dual, complementary forms. Grassmann's effort to achieve recognition resulted, therefore, in an intriguing exploitation of both sides of what has been a fundamental divide in mathematics.

Some accounts of the *Ausdehnungslehre* have assumed that the more 'mathematical' 1862 version appealed to mathematicians and garnered attention from its first appearance, but my account follows more that of Victor Schlegel, Grassmann's colleague, who believed that initially it too suffered the same fate as the earlier version (*Schlegel 1872*, p. vii). The 1862 edition, he judged, failed to give a helpful overview of the whole system and had tiresomely, even if necessarily, long proofs.[6] The 1844 version had gone out of print, with unsold copies being shredded by the publisher, and thus at some point the 1862 edition became per force the principal source for the subject and continued as such to some extent even after the reprint of the earlier version in *1878*.

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[6] Because Felix Klein held Schlegel in an unfavourable light, Schlegel's views are apt not to get the historical weight they deserve. Probably due to Klein's influence, Schlegel was invited to contribute only to the bibliography in the grand *Werke* project (*Grassmann 1894-1911*). On the Schlegel–Klein relationship see *Rowe 1996*.

Hankel is credited with giving the *Ausdehnungslehre* its first major exposure in his extensive and enthusiastic use of it in *Hankel 1867*. He even appears sympathetic to what he terms the ‘philosophical’ presentation of the 1844 version which he deemed appropriate for the subject even if it did turn away some readers (p. 16).

Hankel and Schlegel used the term ‘philosophical’ in the same sense as Grassmann, i.e. referring to the abstract approach to the fundamental entities and operations described above in Section 1. Neither Hankel nor Schlegel seemed to have viewed the pedagogical element, however, as a part of this philosophical aspect, and they made no reference to the larger theme that Grassmann referred to as the scientific nature of the undertaking. These aspects were equally overlooked by those contributors to the development of mathematical logic who read Grassmann, such as E. Schröder, G. Peano and G. Frege. The work that these three read was primarily Grassmann’s arithmetic textbook, the *Lehrbuch der Arithmetik* (*Grassmann 1861*). As remarkably ahead of its time as the *Ausdehnungslehre*, it adopted the same abstract approach from the beginning as the 1844 work, as, for example, by first defining and proving the basic properties for what he termed a ‘fundamental series’ (an infinite cyclic group) before introducing the integers as a series that satisfies the same properties. It made what is generally regarded as the first explicit use in the literature of the principle of mathematical induction and is the source of the theorems and proof structures used in Peano’s presentation of his axioms for the natural numbers.[7] The *Lehrbuch*, however, used only one of the two strands of presentation that Grassmann described in the *Ausdehnungslehre* and made rigor the highest priority. Reviewers at the time generally recognized the advance in rigor the *Lehrbuch* represented, but did not believe that it was suitable for classroom use. (Grassmann’s 1844 *Ausdehnungslehre* was never reviewed.) Frege, in a critical review of arithmetic textbooks of his time, found much to admire in the *Lehrbuch* and only one significant, and easily remedied, flaw—though this was still enough for him to conclude that Grassmann’s ‘rigor’ was thus only superficial (*Frege 1884,* p. 8). Schröder relied upon Grassmann’s *Lehrbuch* in his own textbook of arithmetic and algebra (*1873*) but seems not to have been acquainted with the *Ausdehnungslehre* of 1844. He, in effect, arrived at the same conclusions about the desirability of a dual presentation as Grassmann and used Grassmann’s *Lehrbuch* as an example of a one-sided approach. However appropriate Grassmann’s work was for such an example, this was probably a dismaying state of affairs from Grassmann’s point of view. Schröder’s second chapter began:

> In what follows I will handle the preceding material in two fundamentally different ways which are independent of each other and each of which has its own advantages.

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[7] Grassmann does not give a statement of mathematical induction but rather, in his first use of it in a proof, states that this is an example of a proof by induction. Peano’s system is given in *Peano 1889*, translated in *Peano 1967*. An overview of the *Lehrbuch*’s import is given in *Lewis 1995* and an evaluation of its logical basis is made in *Wang 1957*.

> The so-called independent [*independente*] treatment has the advantage of providing an immediately accessible motivation for complete comprehension …. .
>
> … It is in addition the method through which the theory historically has been gradually developed.
>
> The other method of treatment, which we call the recurrent [*recurrente*], comes from the requirement of greater rigor with a view to achieving the greatest simplification of the initial assumptions and reducing the argument used. We owe this principally to Grassmann and can say that the stated goal is attained through this means; it leaves hardly anything further to be desired with respect to thoroughness. Even if this method may not be recommended for the beginner of average capabilities … it is of such methodological interest that I cannot refrain from assimilating it into the present textbook—especially since the latter aims to be thorough. (*Schröder 1873*, 51-52)

Since Grassmann also used essentially only the one method of presentation in his *1862,* this may have been a period where he simply gave up using the dual method which may have seemed not worth the trouble. Unlike his *Ausdehnungslehre*, arithmetic could not be claimed by Grassmann as a new branch of mathematics and this too may help account for the absence in the *Lehrbuch* of any concern with providing the learner with the same sort of overview as the discoverer. It is not clear what background Grassmann assumed for the *Lehrbuch* but it is likely that the students, if there ever were any who used it, would have had an acquaintance with numbers. In his introduction he emphasized the need for the most rigorous possible approach to inculcate a logical mind-set for the student. 'This goal, however', he stated, 'will not be attained by insisting only on a series of one formula after another without a conceptual development. Rather there must be both: the formulaic development must always proceed hand in hand with the conceptual development' (*Grassmann 1894-1911,* vol. 2, part 1, p. 296). Thus Grassmann was hardly abandoning the basic approach of 1844 but the *Lehrbuch* is decidedly a weaker version in this respect. There is no explicit indication of a change of heart, and it seems unlikely that there would be such a fundamental change given the attachment he expressed for his earlier work in the last years of his life.

Shortly before he died in 1877 Grassmann felt that the rising interest in his work justified a reprint of the 1844 version of the *Ausdehnungslehre*.[8] By 1910 there were a number of purported versions of Grassmann's calculus of extension, from brief 'essentials' in journals to extensive treatises, and from straightforward translations of excerpts to substantially new systems that were nevertheless still in the Grassmann algebraic or analytic geometry tradition. In good part these were motivated by the competition between vectors and W. R. Hamilton's quaternions that has been well described in *Crowe 1985*. By the time the monumental edition of Grassmann's

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[8] *Grassmann 1878* is the basis of the 1844 version printed in vol. I, part 1, of *Grassmann 1894-1911*.

mathematical and physical works, the *Werke* (1894-1911), was completed Grassmann's importance in mathematics was being recognized in the literature. In fact the period from about 1910 to 1940 may have been something of a high point for historical recognition of Grassmann, at least in the area of geometry, even as mathematical interest in him declined during this period. His influence had likely largely faded away by the time of H. G. Forder's *The Calculus of Extension* (*Forder 1941*), which, as the author indicates in the preface, was essentially completed in 1933. The first translation of the 1844 *Ausdehnungslehre* appears to have been into Spanish in *Grassmann 1947*. It is a measure of a hiatus in Grassmann interest that French and English translations, by D. Flament (*Grassmann, H. G. 1994*) and L. Kannenberg (*Grassmann, H. G. 1995*) respectively, appeared so much later. Unfortunately, during this earlier period there seems to have been no publication that delved at all deeply into the unifying theme of the 1844 *Ausdehnungslehre*. 'Unfortunately' because such luminaries as G. Peano, A. N. Whitehead, and E. Cassirer had studied the book with some care and made use of it in various ways.[9] They may not have regarded Grassmann's Schleiermachian-influenced notions as worth attention, but it would have been enlightening to have their critique in any case. The only writers evident from this period who appeared to have an appreciation of this theme were in the United States, and these---Paul Carus and Arthur Schweitzer---did not have such a great international influence in mathematics or philosophy.

Such influence as Carus (1852–1919) had come mainly as the editor of *The Open Court* and *Monist,* journals that were based at La Salle, Illinois, and that were highly regarded in history and philosophy of science circles. Carus had been a student of Grassmann in Stettin and made the *Ausdehnungslehre* a starting point in presenting his own monistic philosophy that sought to unite science and religion.[10] Many of his articles cited Grassmann, and in one, where he gave more detailed attention, he maintained that 'Grassmann has taught us to dive down to the bottom of the problems, where we can understand the origin and whole growth of mathematics and where they are seen to be in connection with the other facts of reality' (*Carus 1889,* p. 1471). In this same paper Carus identifies Grassmann's aim of providing an 'overview' for the learner at each step. Carus gives this an expansive interpretation: 'Grassmann's method allows a survey of the whole field and thus gives to the student that easy freedom which a traveller feels who constantly keeps in sight the point towards which he is journeying, as well as the road on which he approached' (p. 1472).

Schweitzer also spent his career in Illinois, though there does not appear to be any other connection between the two men. Arthur Richard Schweitzer's dissertation in the mathematics department at the University of Chicago, published in 1915, is remarkably relevant to the topic of this paper, but it is also appropriate to dwell on it here because it

[9] The representative works are: *Peano 1888,* translated into English as *Peano 2000*; *Whitehead 1898*, discussed in relation to Grassmann in *Lowe 1985-1990,* vol. 1, pp. 153-56; and *Cassirer 1953,* pp. 96-99.

[10] *Meyer 1962*.

appears not to have been noted in the literature before.[11] Schweitzer (1878–1957) describes the role of the 'leading idea', or '*idée directrice*', in the development of the logic of mathematics. In a short span he draws upon an impressive variety of prominent and not so prominent mathematicians and logicians, from Boole to contemporaries such as C. S. Peirce, B. Russell, D. Hilbert, and H. Poincaré. Grassmann is cited more often than any other person. Schweitzer's *idée directrice* is essentially, he says, Grassmann's *leitende Idee* described above at the beginning of section 2:

> At the beginning, following Grassmann, the leading idea is 'obscure presentiment' [*dunkles Vorgefühl*]; results of this presentiment are then critically analyzed and the discovery of the truth follows if the leading idea is correct. (*Schweitzer 1915,* p. 5)

Schweitzer follows Grassmann in asserting that mathematicians are little disposed to admit *l'idée directrice* into pure mathematics:

> In the midst of this uniformity, Grassmann's expositions form remarkable exceptions. The *Ausdehnungslehre* of 1844 is perhaps unique in the mathematical literature, in the sense that it frequently shows and explicitly recognizes the genetic act of the discovery. (*1915,* p. 13)

As part of his conclusion, Schweitzer gives four examples of leading ideas in the logical foundations of mathematics: the principle of comparison; the principle of continuation; the principle of the economy of thought; and the principle of the special situation. One example of the first principle is E. H. Moore's dictum in his *General analysis* of 1910 that 'The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features' (*Moore 1935,* p. 1). The *idée directrice* stands as an instrument in Schweitzer's scheme mediating between a class of givens on one side and the actual implementing agents on the other. The latter he describes as 'explicit mediators', *médiateurs explicites*, and one example is Moore's general analysis as mediating between four mathematical theories. The *Ausdehnungslehre* not unexpectedly takes pride of place as being the premier historical example with explicit mediations—falling mainly under the principle of comparison—permeating the work. Schweitzer even goes so far as to suggest that Grassmann may have aimed too high in this direction when he stated that the theory of extension 'could be said to form the keystone of the entire structure of mathematics' (quoted above in Section 1 from Grassmann's *1862*):

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[11] In *Schweitzer 1915* the author mentions that Schleiermacher is undoubtedly a source of a number of Grassmann's notions, in particular that of '*leitende Idee*'. Though he does not justify this observation beyond citing a few parallel passages in Grassmann and Schleiermacher, it is the closest statement I know of to the thesis in my *1977* in prior literature.

> Grassmann recognized clearly that mathematical theories were only instruments; in fact, he even went so far as to believe that his calculus was of absolute universality in mathematics. This belief was probably erroneous; perhaps Grassmann had in mind the possibility of incorporating symbolic logic in his system. (p. 21)

There appears to be minimal information about Schweitzer in standard sources.[12] He published a number of works that reflect a University of Chicago background, especially in geometry (O. Veblen, E. H. Moore), as well as a background in the US school of pragmatism (C. S. Peirce, W. James, J. Dewey, G. H. Mead, A. W. Moore). It seems clear, however, that his main inspiration comes from Grassmann: some of his publications relate very directly to Grassmann, such as *Schweitzer 1908* while others, such as *Schweitzer 1909* and *Schweitzer 1913* frequently cite Grassmann. A piece intriguingly titled 'The Logic of Grassmann's Extensive Calculus', gives a collection of axioms gathered from the 1844 *Ausdehnungslehre* from which, he proposes, the results in the 1862 edition and in Grassmann's subsequent journal articles can be proven. The fact that Schweitzer is so little cited in the literature of his time probably indicates that his Grassmannian direction was regarded as out of the mainstream of mathematics and that the *Ausdehnungslehre* was deemed to have served its purpose by then. Thus its use as an authoritative mathematical work from which to draw examples, for instance to refute the mathematical contentions of the US school of 'new realists' as Schweitzer undertook to do, may have been regarded as dated at the time.[13]

4. A revival of interest

In 1994 the sesquicentennial of the 1844 edition of Hermann Grassmann's *Ausdehnungslehre* was observed on the island of Rügen, not far from the Baltic port city of Sczcezin in Poland where Grassmann spent most of his life. (It was named Stettin in Grassmann's time when it was a part of Germany.) This gathering brought together many threads of Grassmann's life as mathematician, physicist, philologist, teacher, and musicologist. It testified to a continuing interest in his work and to an ongoing influence of that work. As such anniversary celebrations can do, this one suggested new avenues of possible influence and new insights into Grassmann's genius. It also gave a vantage point from which to look anew at Grassmann as a whole and to explore to what extent he had a unified program across the various disciplines to which he contributed. This is probably the best source, next to the *Werke*, for anyone wishing to find out more about Grassmann.

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[12] Schweitzer's birth date is given as 1877 in *Sommerville 1970* but appears as 1878 in the Library of Congress catalogue. There is a notice in the *American Mathematical Monthly,* 64(1957), p. 611, of his death on 12 June 1957.

[13] *Schweitzer 1914* is reprinted in *de Waal 2001* from which I first learned of this Grassmannian.

Part of the new interest in Grassmann may stem from the increased importance of linear algebra. The reason for the latter has been summed up by the US mathematician Alan Tucker:

> In the 1960s, linear algebra was positioned to be the first real mathematics course in the undergraduate mathematics curriculum in part because its theory is so well structured and comprehensive, yet requires limited mathematical prerequisites. A mastery of finite vector spaces, linear transformations, and their extensions to function spaces is essential for a practitioner or researcher in most areas of pure and applied mathematics. Linear algebra is the mathematics of our modern technological world of complex multivariable systems and computers. (*Tucker 1993,* p. 3)

The trend Tucker describes is exemplified by Jean-Luc Dorier, who was led to pursue the history of linear algebra by his researches into the teaching of vector space theory at French universities. Given this background it seems especially appropriate that he would appreciate the pedagogical aspects of Grassmann's *Ausdehnungslehre*. In *Dorier 1996* he analyzed a principal theorem dealing with the dimension of a vector space which Grassmann stated as:

> The same system of *m*-th order is generable by any *m* methods of generation belonging to it that are mutually independent (in the sense of §16), that is, that are included in no system of lower order (than the *m*-th). (*Grassmann 1844,* §20)

Grassmann has not yet introduced numbers as multipliers of his generating entities and thus the concept of a linear combination of them becomes possible only later in his presentation. Nevertheless in modern terms, taking Grassmann's statement in the context of the whole *Ausdehnungslehre*, "methods of generation … that are mutually independent" corresponds to a basis of the *m*-dimensional space or "system of *m*-th order." Part of the importance of the concept of basis is that any two bases for the same space contain the same number of elements (i.e. base vectors or "methods of generation") and Grassmann's statement above is a lemma used in proving this property. As a whole it corresponds to the following modern form, taking V as a "system of *m*-th order" (F, the field of numbers associated with V, has no direct correspondence in Grassmann at this point):

> If $v_1, \ldots, v_m$ is a basis of V over F and if $w_1, \ldots, w_n$ in V are linearly independent then $n \leq m$. (*Herstein 1964,* p. 140, interchanging m and n)

Grassmann recognized the same two principal methods for proving this that can be found in modern textbooks: an elimination method and the exchange method. He chose the exchange method. (This procedure, by the way, is often named after Steinitz, who used it in the second decade of the twentieth century with no mention of Grassmann. The name exchange theorem, *Austauschsatz*, came later.)

In the elimination method one might show that every subset of V which contains more than m vectors is linearly dependent. In the exchange method the basis of v's is augmented with w_n and it is argued that since this resulting set spans the space and is linearly dependent it contains a subset which is a basis. Sufficient of the v's are then removed to leave only this basis, w_{n-1} is adjoined to them to form a new linearly dependent, spanning set, and, applying the same argument as before, v's are removed from this set to arrive at a new basis. This process is repeated as far as w_2. This 'exchange' of w's for v's cannot proceed to replace all the v's since the remaining w_1 cannot be expressed as a linear combination of the w's alone. Thus $n \leq m$. This précis of a modern proof is far removed from Grassmann's presentation—though he has all of the concepts, he has little of the compact terminology that has since been developed.

One of the reasons many modern authors tend to favour the exchange method of proof for this theorem probably is that it uses essentially only concepts and results that have been developed immediately before, rather than, for example, using coordinates and converting it into a problem dealing more with systems of linear equations. Also, it is a method that can be used in other contexts. Dorier, after careful analysis of the way Grassmann used this method in both versions of the *Ausdehnungslehre*, came to the conclusion that Grassmann developed the method precisely as a consequence of his pedagogical philosophy and that it was only in the version in the *Ausdehnungslehre* of 1844 that the justification of the method is completely evident. As Grassmann himself stated, it was not only 'elementary, but in addition has the advantage that the most essential basic relations between the extensive magnitudes stands out more clearly' (*Grassmann 1862, §24*). Dorier agreed that the use of the exchange procedure 'comes naturally within the form of presentation adopted in the *Ausdehnungslehre*' (p. 182).

The teaching of linear algebra has been the subject of a different type of study in *Harel 1999*. Though Grassmann and the history of linear algebra are not involved, this paper gives an interesting corroboration from a student's point of view of the two-sided nature of learning. Based on classroom observations, interviews, and written tests, he identifies two main categories of student understandings of proofs. In one there appears to be a reliance on spatial imagery to the extent that students do not see geometric properties as abstract structures. The second, experienced by more advanced students, is the questioning of the truth of a theorem even after apparently understanding all the steps of its proof. These two groupings probably confirm what an experienced teacher would have conjectured, but as the author points out, these understandings have parallels in the historical development of mathematics itself: he identifies the first---need for spatial imagery---with the Greek period and the second---need to know 'why' rather than just 'how'---with the sixteenth and seventeenth centuries when some mathematicians, such as Descartes, rejected indirect proofs in favour of direct demonstrations of mathematical truths.

5. Revising Grassmann: points or vectors?

Several commentators have noticed what might be taken as a difficulty, if not a flaw, in Grassmann's handling of the notion of the difference of two points, $B - A$. Grassmann's complete discussion is quite general, but even if we restrict ourselves to the example of points in a Euclidean line, plane, or three-dimensional space, this innocentsounding issue raises some deep problems of interpretation of Grassmann's text. This in turn raises different possibilities for re-expressing Grassmann's ideas for a textbook of today. The two principal examples given here are the US mathematicians A. Swimmer and F. W. Lawvere.

Alvin Swimmer points to the 'miscegenated addition' in modern treatments of affine space which give a meaning to adding a point and a vector, resulting in a point, or to subtracting one point from another, resulting in a vector (*Swimmer 1996*). There is no meaning to adding one point to another. He claims that this 'unfortunate' situation can be traced back to the status of vectors, rather than points, as the fundamental entity in the *Ausdehnungslehre*. It could be argued that historically Grassmann was not to blame for this; to the extent that he was a source it is likely to have come from a particular interpretation that gained currency.[14] However, it is not Swimmer's purpose to back up this claim, but rather to show how one might obtain the best of several worlds (physical, algebraic and geometrical) by recasting Grassmann's presentation. This is done by following Möbius's 1827 barycentric calculus, an influence on Grassmann, more closely than did Grassmann. Starting with weighted points we can define the sum of two points as their centre of mass. The difference, $B - A$, when the two weights are each equal to 1, is a point at infinity. Points at infinity can be distinguished by their magnitudes; in the case of $B - A$ this is the length of the oriented line segment from A to B. $B - A$ is also associated with a family of parallel lines. 'Thus "vector" and "point at infinity" are simply two different ways of looking at the same concept' (p. 274). This results in an introduction to the subject which indeed exhibits a unity that appears to be lacking in the *Ausdehnungslehre* of 1844 and still satisfies Grassmann's aim of presenting matters in such a way that 'the intellect grasps the progressive development of the idea with each formal development of the mathematics' (*1894,* p. 9, Swimmer's translation).

William Lawvere, in his *1996* addresses the same issue of points versus vectors from a different perspective. As one of the principal proponents of category theory, it is noteworthy that he regards Grassmann as a precursor of that subject. In addition to a certain similarity of approach to the development of algebraic structures, the *Ausdehnungslehre* and category theory raised analogous metamathematical issues in their respective times, since each offered not only a new way of presenting results but also the possibility of a new foundation of mathematics. When it made its main debut into the textbook world through *Mac Lane and Birkhoff 1967,* category theory stirred an interest with a touch of controversy over its importance for mathematics as a whole. Lawvere's paper, however, is concerned with showing that viewing the *Ausdehnungslehre* from a functorial standpoint brings out a greater richness of structure

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[14] In defence of Grassmann it may be possible to argue that his more fundamental notion is 'change' (*Änderung*) and that this would be needed to distinguish whatever entities, either points or vectors (or numbers as in the *Lehrbuch der Arithmetik)*, were being generated. Admittedly 'change' and 'vector' are naturally close concepts but if it is possible to separate them, Swimmer's own construction shows that one need not take vector as the initially generated entities within Grassmann's general framework.

than, for one thing, the classical exterior algebra that derives from it. The difference $B - A$ is seen as the result of the boundary operator acting on the product *AB* (the axial vector from *A* to *B*): $\partial(AB) = B - A$. Lawvere proposes that this operator is the same as Grassmann's *Ausweichung* or divergence. Instead of looking for a '+', i.e. the operator of which this is the inverse, he suggests conceiving the relationship between the operators as one of 'adjointness, in the sense of category theory' (p. 257). It has been something of a specialty of Lawvere's to identify adjointness in a wide range of fields, including mathematical logic, as in *Lawvere 1969*. This particular example has an interesting parallel in a remark, not noted here by Lawvere, in a summary of Grassmann's work raising an objection to Grassmann's simply substituting the vector (*Strecke*) for what we have been writing as $B - A$. (*Sturm/Schröder/Sohnke 1879,* p. 7.) It perhaps would have been better, they maintain, for Grassmann to have identified this with a fixed direction, or with its fixed infinitely distant point, and to adjoin (*adjungiren*) the vector.

The contributions of Swimmer and Lawvere (the latter also acknowledges the collaboration of his colleague Stephen H. Schanuel) should inform all future readings of the *Ausdehnungslehre*. Though they develop their own programs, their work sheds light on Grassmann's work by suggesting new understandings of it. Furthermore, they exemplify Grassmann's meaning of 'scientific quality', that calls for focusing on the learner, in a modern context. They consider it an important and natural part of their business to write texts for students, not just to convey the subject matter but also to demonstrate that their approaches are eminently suited for study and for serving as gateways to further mathematical avenues. (*Lawvere 1996,* p. 256*; Swimmer 1996,* p. 279.)

6. Conclusion

Most mathematicians would probably agree that it is desirable to present their subject, at least for some purposes such as textbooks, in a dual fashion that combines rigor with what Grassmann called an 'overview'. Even in a research paper a practical advantage can be recognized: an error that leaves a hole that is deemed fatal in a proof may be evident as an easily reparable slip in a presentation that gives the reader more explanation and context. Grassmann held a strong version of the thesis, namely that this method of dual presentation is as much a requirement of mathematics as its logical structure. It would presumably follow from this that perfecting such a method is as much the job of a mathematician as getting the proofs right. Regardless of whether this rather radical view has ever been subscribed to by others or not, it leads to questions that are of perennial interest to educators. Should a research paper, even one that amounts to no more than giving a different proof of an already established result, be presented following the same general principles that the author would use in a text for students? It might seem reasonable to do so, since in both cases it is assumed that the reader, whatever level their background, has not seen the 'new' material. If it is a learning process in both cases, is that process essentially different for a school student and a research mathematician? How do we acquire knowledge?

The aim of this paper is to contribute to a better historical understanding of Grassmann, but this in turn may inform discussions about theories of learning mathematics. Admittedly we do not have much more on Grassmann's side to help in such a comparison beyond the sources included in the bibliography to this paper. It would be useful to know what Grassmann did in the classroom on a daily basis, and unfortunately this is not clear. There is no extant record by Grassmann, and those few of his students that we have reports from agree that he was a pleasant person and a devoted teacher but do not convey a picture of how he might have put his textbooks into practice. Paul Carus, the one student who has written on Grassmann's ideas, relies entirely on his *Ausdehnungslehre* rather than on direct personal contact.

There are few clues in Grassmann's personal background. His main academic training was directed at becoming a Protestant minister, and he studied languages and philosophy at university. He learned mathematics on his own and from his father, a schoolmaster in Stettin. Grassmann followed his father into the same profession in the same city. Contrary to some accounts, it should not be assumed that Grassmann was thus an overburdened and frustrated secondary-school teacher. Though he did aspire to a university position, this was mainly for the purpose of getting into a more academic environment: the university positions he applied for would not necessarily have given him more pay or more research time than the prestigious and well-endowed school at which he taught.[15] An invariant theme in his life was thus teaching, or more generally the presentation of mathematics for the benefit of others. His mathematical creativity, far from being compromised, appears to have been motivated in large measure by this calling.

[15] On the typical German university compared with a gymnasium, such as that Grassmann taught in, see *Rowe 1996,* pp. 132-34. For descriptions of Grassmann as a teacher and of his other textbooks in trigonometry, Latin, and German, see the index to the proceedings *Schubring 1996b*.

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